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Counting Form Factors of Twist-Two Operators

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Abstract

We present a simple method to count the number of hadronic form factors based on the partial wave formalism and crossing symmetry. In particular, we show that the number of independent nucleon form factors of spin-n, twist-2 operators (the vector current and energy-momentum tensor being special examples) is $n+1$. These generalized form factors define the generalized (off-forward) parton distributions that have been studied extensively in the recent literature. In proving this result, we also show how the J^{PC} rules for onium states arise in the helicity formalism.

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The generalized (off-forward) parton distributions of hadrons, the nucleon in particular, have attracted considerable theoretical and experimental interest in the last few years [1]. These distributions generalize the well-known Feynman parton distributions as well as the elastic electromagnetic form factors. Most interestingly, the distributions contain information about the orbital motion of partons in a polarized nucleon. One can, for instance, deduce the amount of the nucleon spin carried by the quark orbital angular momentum once these distributions are known [2].

One way to define the generalized parton distributions is to consider the nucleon matrix elements of the twist-2 operators. An example is the chiral-even spin-independent quark operators

$$\hat{O}^{\mu_1\cdots\mu_n} = \overline{\psi}(0)\, i\overset{\leftrightarrow}{D}^{(\mu_1} \cdots i\overset{\leftrightarrow}{D}^{\mu_n)}\psi(0) \quad n = 1, 2, ... \;, \tag{1}$$

where $\overset{\leftrightarrow}{D}^{\mu_1} = (\overset{\leftarrow}{D}^{\mu_1} - \vec{D}^{\mu_1})/2$, which form totally symmetric tensor representations of the Lorentz group when the Lorentz indices $\mu_1, \cdots, \mu_n$ are symmetrized and rendered traceless (indicated by parentheses). Clearly, this tower of operators is a generalization of familiar vector current $\bar{\psi}\gamma^\mu\psi$. The forward matrix elements of the above operators define the *generalized charges*, a_n

$$\langle P|\hat{O}^{\mu_1\cdots\mu_n}|P\rangle = 2a_n(Q^2)P^{\mu_1}\cdots P^{\mu_n} \;, \tag{2}$$

where Q^2 is a renormalization scale. The nucleon state is normalized covariantly: $\langle P'|P\rangle = (2P^0)(2\pi)^3\delta^3(\vec{P} - \vec{P}')$. Feynman's unpolarized quark distribution $q(x, Q^2)$ can be obtained directly from the generalized charges,

$$\int_{-1}^{1} x^{n-1}q(x, Q^2)dx = a_n(Q^2) \;, \tag{3}$$

with $n \geq 1$. As usual, $q(x, Q^2)$ at negative x represents the distribution of antiquarks.

The off-forward matrix elements of the twist-2 operators define *generalized form factors* $A_{n,2i}(t, Q^2)$, $B_{n,2i}(t, Q^2)$ and $C_n(t, Q^2)$ [1],

$$\begin{aligned}
\langle P'|\hat{O}^{\mu_1\cdots\mu_n}|P\rangle = \quad & \overline{U}(P')\gamma^{(\mu_1}U(P)\sum_{i=0}^{[\frac{n-1}{2}]} A_{n,2i}(t, Q^2)\, q^{\mu_2}\cdots q^{\mu_{2i+1}}\overline{P}^{\mu_{2i+2}}\cdots\overline{P}^{\mu_n)} \\
& + \overline{U}(P')\frac{\sigma^{(\mu_1\alpha}iq_\alpha}{2M}U(P)\sum_{i=0}^{[\frac{n-1}{2}]} B_{n,2i}(t, Q^2)\, q^{\mu_2}\cdots q^{\mu_{2i+1}}\overline{P}^{\mu_{2i+2}}\cdots\overline{P}^{\mu_n)} \\
& + \mathrm{mod}(n+1, 2)\ \frac{1}{M}\bar{U}(P')U(P)\ C_n(t, Q^2)\, q^{(\mu_1}\cdots q^{\mu_n)} \;,
\end{aligned} \tag{4}$$

where $\overline{U}(P')$ and $U(P)$ are the Dirac spinors and $\mathrm{mod}(n+1, 2)$ is 1 when n even, 0 when n odd. The four-momentum transfer is denoted by $q = P' - P$, $t = q^2$, and $\overline{P} = (P' + P)/2$. For $n \geq 1$, even or odd, there are exactly $n + 1$ form factors. The moments of the off-forward parton distributions $H(x, \xi, t)$ and $E(x, \xi, t)$ [1] are related to linear combinations of the above form factors.

The key question we address in this short paper is why there are exactly $n + 1$ form factors. We anticipate that the answer also addresses the general question of how to count the number of form factors of any operator between hadron states.

To motivate a simple counting procedure, we first count the helicity amplitudes in the Breit frame. We start with the simplest example: the vector current $\hat{O}^\mu = \bar{\psi}\gamma^\mu\psi$, which is known to define two form factors for the nucleon states,

$$\langle P'|\hat{O}^\mu|P\rangle = \overline{U}(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]U(P)\ . \tag{5}$$

To understand why there are two, consider the Breit frame, in which the initial and final nucleon have 3-momenta of the same magnitude but opposite directions. The 4-vector current can be decomposed into a 3-vector source plus a 3-scalar source. Because of current conservation, the scalar degree of freedom does not provide independent information. Thus, one need only count the number of independent helicity amplitudes for $P +$ (3-vector source) $\to P'$. Because of the collinearity of the process in this frame, the angular momentum projection along the direction of the reaction is conserved. Hence, there are four possible helicity amplitudes: $A_{1/2,0\to-1/2}$, $A_{-1/2,0\to1/2}$, $A_{1/2,1\to1/2}$, and $A_{-1/2,-1\to-1/2}$, where the indices label the helicities of the initial nucleon, the source, and the final nucleon, respectively. However, amplitudes that differ only by flipping the signs of all helicities are related through parity invariance, and therefore only two of the amplitudes are independent, as expected.

A subtlety arises if the vector current is not conserved. Then one has an extra spin-0 source that leads to a new helicity amplitude $A_{1/2,0'\to-1/2}$ (and its parity partner), suggesting a total of three independent form factors. It turns out, however, that if the (hermitian) vector current transforms like $\bar{\psi}\gamma^\mu\psi$ under time reversal, the new amplitude vanishes due to time-reversal symmetry. Indeed, let us write down possible invariant form factors in this case,

$$\langle P'|\hat{O}^\mu|P\rangle = \overline{U}(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M} + F_3(q^2)iq^\mu\right]U(P)\ , \tag{6}$$

where according to hermiticity we have included a factor of i in front of q^μ so that $F_3(q^2)$ is real: Since hermitian conjugation exchanges bra and ket, then $P \leftrightarrow P'$ in the external states, and the sign of $q = P - P'$ is flipped (Note that this requirement only holds if the initial and final particles are the same). On the other hand, time reversal symmetry requires

$$\begin{aligned}\langle P'|\hat{O}^\mu|P\rangle &= \langle P'|T^{-1}T\hat{O}^\mu T^{-1}T|P\rangle^* \\ &= \overline{U}(P')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M} - F_3(q^2)iq^\mu\right]U(P)\ ,\end{aligned} \tag{7}$$

where in the second line we use $T\hat{O}^\mu T^{-1} = \hat{O}_\mu$ and the time-reversal transformation of the states. Therefore $F_3(q^2)$ must vanish, and we again only have two form factors.

The above example shows that helicity counting in the Breit frame cannot provide the correct answer unless time-reversal symmetry is taken into account. This is easy to do for helicity amplitudes in elastic scattering. For form factors, however, the form of the constraint is less clear. One might follow the above approach by writing down all possible invariant form factors and imposing time-reversal symmetry: One then obtains Eq. (4). Unfortunately, this procedure does not provide a simple way to understand the physics underlying the counting.

In the remainder of the paper, we present an alternative counting method. We first recall a basic property of relativistic quantum field theory that the number of independent amplitudes is the same in all crossed channels. Then we count the number of independent matrix elements, $\langle P\overline{P}|\hat{O}^{\mu_1\cdots\mu_n}|0\rangle$, corresponding to $P\overline{P}$ creation from the twist-2 source. In the center of momentum (c.m.) frame, the enumeration of possible $P\overline{P}$ states is well-known: Since $S=0,1$, $\vec{J}=\vec{L}+\vec{S}$, $P=(-1)^{L+1}$, and $C=(-1)^{L+S}$, the list of allowed $J^{PC}(L)$ reads

$$\begin{array}{llll} 0^{++}(1), & 0^{-+}(0), & & \\ 1^{++}(1), & 1^{+-}(1), & 1^{--}(0), & 1^{--}(2), \\ 2^{++}(1), & 2^{++}(3), & 2^{-+}(2), & 2^{--}(2), \\ 3^{++}(3), & 3^{+-}(3), & 3^{--}(2), & 3^{--}(4), \\ \cdots . & & & \end{array} \tag{8}$$

For each $J \geq 1$, there are four possible states. Two of them have the same J^{PC} but different L: $J^{(-1)^J,(-1)^J}$, with $L = J \pm 1$.

One might suspect that the classification of $P\overline{P}$ into states of definite J^{PC} using $\vec{L}$ (orbital angular momentum being defined in the $P\overline{P}$ c.m. frame) and $\vec{S}$ (each nucleon spin being defined in its own rest frame) is inherently non-relativistic. This is not so because the relativistic spin wave functions transform the same way under space rotations as their non-relativistic counterparts [3]. The fully relativistic treatment [4] employing the helicity formalism gives exactly the same rules for allowed J^{PC} with the same multiplicity of amplitudes, as we now show.

Let $\psi_{JM\lambda_1\lambda_2}$ be a two-particle helicity state. With $\eta_{1,2}$ being the intrinsic parities of the two particles, the action of parity gives

$$\hat{P}\psi_{JM\lambda_1\lambda_2} = \eta_1\eta_2(-1)^J\psi_{JM-\lambda_1-\lambda_2} \ , \tag{9}$$

a property we used above in reducing the number of helicity amplitudes. The factor $\eta_1\eta_2$ is unity for any particle-antiparticle pair. Similarly, the action of charge conjugation on such a self-conjugate pair gives

$$\hat{C}\psi_{JM\lambda_1\lambda_2} = (-1)^J\psi_{JM\lambda_2\lambda_1} \ , \tag{10}$$

Thus, suppressing the JM subscripts, one may form states of definite J^{PC}:

$$\begin{array}{ll} PC=++: & \psi_{\lambda_1\lambda_2}+(-1)^J\psi_{\lambda_2\lambda_1}+(-1)^J\psi_{-\lambda_1-\lambda_2}+\psi_{-\lambda_2-\lambda_1} \ , \\ PC=+-: & \psi_{\lambda_1\lambda_2}-(-1)^J\psi_{\lambda_2\lambda_1}+(-1)^J\psi_{-\lambda_1-\lambda_2}-\psi_{-\lambda_2-\lambda_1} \ , \\ PC=-+: & \psi_{\lambda_1\lambda_2}+(-1)^J\psi_{\lambda_2\lambda_1}-(-1)^J\psi_{-\lambda_1-\lambda_2}-\psi_{-\lambda_2-\lambda_1} \ , \\ PC=--: & \psi_{\lambda_1\lambda_2}-(-1)^J\psi_{\lambda_2\lambda_1}-(-1)^J\psi_{-\lambda_1-\lambda_2}+\psi_{-\lambda_2-\lambda_1} \ . \end{array} \tag{11}$$

For spin-1/2 particles, the only independent choices are $\lambda_1=\lambda_2$ and $\lambda_1=-\lambda_2$. In the former case, Eqs. (11) show that only the two amplitudes with $C=(-1)^J$ (and either parity) are nonvanishing, while in the latter case only the two amplitudes with $PC=+1$ survive. These are the allowed sequences $0^{++},0^{-+},1^{+-},1^{--},2^{++},2^{-+},\cdots$, and $1^{++},1^{--},2^{++},2^{--},\cdots$, respectively. Note that the amplitudes 0^{++} and 0^{--} have not been included in the second list, since their amplitudes according to Eq. (11) would be $\psi_{\lambda-\lambda} \pm \psi_{-\lambda\lambda}$; however, since the particles are back-to-back in the c.m., the existence of each term requires an angular momentum projection along the axis of $+1$ or -1, which is forbidden since $J=0$. Hence, the 0^{++} amplitude with $\lambda_1=-\lambda_2$ and the 0^{--} amplitude vanish. Thus, the complete set of forbidden states (the so-called "exotics" in the context of the quark model) is, precisely as in the non-relativistic counting, $0^{--}, 0^{+-}, 1^{-+}, \cdots, J^{(-1)^J,(-1)^{J+1}}, \cdots$. Note also that the amplitudes $J^{(-1)^J,(-1)^J}$ appear twice in the "non-exotic" list (except 0^{++}, which only appears once), exactly as in the non-relativistic case.

Clearly, only an external source with the same J^{PC} can produce these states. Let us classify J^{PC} of the twist-2 operators. $\hat{O}^{\mu_1\cdots\mu_n}$ furnishes an $(n/2, n/2)$ representation of Lorentz group [3] and has $(n+1)^2$ independent components: Under spatial rotations, the tensor is decomposed into the angular momentum components $J^P = 0^+, 1^-, \cdots, n^{(-1)^n}$, each with natural parity $(-1)^J$. The charge-conjugation parity is clearly $(-1)^n$, a sign appearing for each of the n Lorentz indices. Thus, the J^{PC} content of the twist-2 operator is

$$\begin{aligned}
&n=0, \quad 0^{++}, \\
&n=1, \quad 0^{+-}, \quad 1^{--}, \\
&n=2, \quad 0^{++}, \quad 1^{-+}, \quad 2^{++}, \\
&n=3, \quad 0^{+-}, \quad 1^{--}, \quad 2^{+-}, \quad 3^{--}, \\
&\cdots .
\end{aligned} \tag{12}$$

Now counting the number of independent matrix elements is straightforward. Since the use of L may be more familiar to the reader, and gives the same counting as the helicity formalism, we retain the L labels for convenience. For $n=1$, only the 1^{--} source is effective. According to Eqs. (8), it can create two 1^{--} states (with $L=0$ and 2), and hence there are two independent matrix elements. For $n=2$, both 0^{++} and 2^{++} sources are effective. While 0^{++} can only create one state, the 2^{++} source can create two independent states (with $L=1$ and 3). Therefore, there are three independent matrix elements. A list of matrix elements in terms of the quantum numbers reads

$$\begin{aligned}
&n=0, \quad J^{PC}(L) = 0^{++}(1), \\
&n=1, \quad J^{PC}(L) = 1^{--}(0), \quad 1^{--}(2), \\
&n=2, \quad J^{PC}(L) = 0^{++}(1), \quad 2^{++}(1), \quad 2^{++}(3), \\
&n=3, \quad J^{PC}(L) = 1^{--}(0), \quad 1^{--}(2), \quad 3^{--}(2), \quad 3^{--}(4), \\
&\cdots .
\end{aligned} \tag{13}$$

This simple pattern is easily extended and proves that there are $n+1$ independent matrix elements for arbitrary $n \geq 0$, implying $n+1$ form factors in $\langle P'|\hat{O}^{\mu_1\cdots\mu_n}|P\rangle$.

The method presented above is completely general. One can use it to count the form factors of a general operator in any hadron states. The procedure is summarized here again: First one goes to the crossed channel in which the operator serves as a source for creating a particle-antiparticle pair. Then allowed J^{PC} values for the operator and particle-antiparticle pair are enumerated and matched. Crossing symmetry plus C and P invariance replaces the use of T invariance in the direct channel. Finally, for each J^{PC}, the number of form factors is determined by the number of independent amplitudes for the creation process. For example, it is easy to show by the same technique that the number of independent form factors of the twist-2 operators in a spinless state like the pion is $[n/2]+1$: Here, Eq. (11) shows that only the series $0^{++}, 1^{--}, 2^{++}, \cdots$ occurs, each of these amplitudes appearing only once.

To summarize, we have presented a simple and general method to count the number of form factors. The method provides a straightforward verification that the number of form factors of a twist-2, spin-n operator is $n + 1$ for nucleon states.

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